MATHEMATICAL PHYSICS

BY

DR. UPAKUL MAHANTA,

Asst.Prof., Department of Physics,
Bhattadev University

BSc. 4th Semester Class notes

Contents

Chapter 1

Legendre Differential Equation (LDE)

Dr. Upakul Mahanta, Department of Physics, Bhattadev University

1.1 Introduction

This differential equation is named after Adrien-Marie Legendre. This ordinary differential equation is frequently encountered in physics and other technical fields. In particular, it occurs when solving Laplace's equation (and related partial differential equations) in spherical coordinates. Although the origins of the equation are important in the physical applications, for our purposes here we need concern ourselves only with the equation itself.
The LDE is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{1.1}$$

$$\frac{dy}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \tag{1.2}$$

1.2 Properties of LDE:

1)Introduced in 1784 by the French mathematician A. M. Legendre$(1752-1833)$.
2)The Legendre polynomials occur whenever you solve a differential equation containing the Laplace operator in spherical coordinates. Since the Laplace operator appears in many important equations (wave equation, Schrdinger equation, electrostatics, heat conductance), the Legendre polynomials are used all over physics.
3) It has regular singular points at $x=-1$ and at $x=1$
4) Thus it has a range of points [-1,1] including 0 as regular singular points.
5) The term $n(n+1)$ is the eigen values of the LDE.
5) The solution for LDE can be expanded both as ascending (like Frobenius method) and decending powers of x from mathematical analysis point of view.
6) But in physics the decending power of x expansion finds application.
7)Legendre functions are important in problems involving spheres or spherical coordinates. Due to their orthogonality properties they are also useful in numerical analysis.
8)Also known as spherical harmonics or zonal harmonics. Called as *Kugelfunktionen*in German. (Kugel in German means Sphere).
Let us now analyse the solution for the LDE. As we doing mathematical physics therefore assume the series solution of LDE as decending order

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k x^{m-k} \\ y' &= (m-k)\sum_{k=0}^{\infty} a_k x^{m-k-1} \\ y'' &= (m-k)(m-k-1)\sum_{k=0}^{\infty} a_k x^{m-k-2} \end{aligned}$$

Next time onwards I am not gonna write the summation sign as well as its limits. Because it sucks time. But you are requested to keep that in mind even if I am not writing that it is always there. Let us now back substitute the above equations in the original LDE and do some little bit of normal algebra.

$$\begin{aligned}
(1-x^2)(m-k)(m-k-1)a_k x^{m-k-2} - 2x(m-k)a_k x^{m-k-1} + n(n+1)a_k x^{m-k} &= 0\\
(m-k)(m-k-1)a_k x^{m-k-2} - (m-k)(m-k-1)a_k x^{m-k} - 2(m-k)x^{m-k} + n(n+1)a_k x^{m-k} &= 0\\
(m-k)(m-k-1)a_k x^{m-k-2} - (m-k)a_k x^{m-k}[(m-k-1)+2] + n(n+1)a_k x^{m-k} &= 0\\
(m-k)(m-k-1)a_k x^{m-k-2} - a_k x^{m-k}[(m-k)(m-k+1)] + n(n+1)a_k x^{m-k} &= 0\\
(m-k)(m-k-1)a_k x^{m-k-2} + a_k[n(n+1)-(m-k)(m-k+1)]x^{m-k} &= 0
\end{aligned}$$

The last equation is an identity and to get the indicial equation we generally equate the co-efficient of lowest power of x to 0 and in there we will also put $k = 0$. But since we have expanded the series in decending power of x we therefore will do the reverse. That is we are going to equate the co-eff. of the highest power of x to 0. And it's not very tough to identify that. Yes, it's x^{m-k}.

$$a_0[n(n+1)-(m-0)(m-0+1)] = 0 \tag{1.3}$$

Since $a_0 \neq 0$ therefore

$$\begin{aligned}
n(n+1) - m(m+1) &= 0\\
n^2 + n - m^2 - m &= 0\\
(n-m)(n+m+1) &= 0
\end{aligned}$$

Thus the roots of the indicial equations are either $m = n$ or $m = -n-1$
Lets us now find the connection between the various order co-efficients. Since again we working on decending powers of x expansion (unlike Frobenius) therfore we will be lowering the all higher powers of x to the lowest powers x. And that can easily be done by replacing all the "k"s by "$k+2$"s. Thus equ^n(10) will yield

$$\begin{aligned}
(m-k)(m-k-1)a_k + a_{k+2}[n(n+1)-(m-(k+2))(m-(k+2)+1)] &= 0\\
(m-k)(m-k-1)a_k + a_{k+2}[n(n+1)-(m-k-2)(m-k-1)] &= 0
\end{aligned}$$

Thus

$$a_{k+2} = \frac{(m-k)(m-k-1)}{n(n+1)-(m-k-2)(m-k-1)}a_k \tag{1.4}$$

Case I: When $m = n$

We can simplify denominator of the last expression putting $m = n$ down to the following

$$a_{k+2} = -\frac{(n-k)(n-k-1)}{(2n-k-1)(k+2)}a_k \tag{1.5}$$

To get the co-eff. a_1 we have to put the value of k equal to -1 in the above expression. But we know that the values of k runs from 0 to ∞ ie -ve values of k do not exsist which means $a_{-1} = 0$. That will in turn imply

$$a_1 = a_3 = a_5 = ... = 0$$

Which means all the odd co-eff. will be 0
now if we put $k = 0$ then we get the connection between a_2 and a_0. ie

$$k = 0, \qquad a_2 = -\frac{n(n-1)}{(2n-1)2}a_0 \tag{1.6}$$

we now put $k = 2$ to get the relationship between a_4 and a_2. But then a_2 is connected to a_0. That means a_4 can be related to a_0.

$$k = 2, \quad a_4 = -\frac{(n-2)(n-3)}{(2n-3)4}a_2 = -\left[\frac{(n-2)(n-3)}{(2n-3)4}\right]\left[-\frac{n(n-1)}{(2n-1)2}\right]a_0 = \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)4.2}a_0 \tag{1.7}$$

In the similar notion

$$k = 4, \quad a_6 = -\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5)6.4.2}a_0 \tag{1.8}$$

Thus we can expand the solution as

$$\begin{aligned}
y =& a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + ...\\
y =& a_0 + \frac{n(n-1)}{(2n-1)2}a_0x^2 + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)4.2}a_0x^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5)6.4.2}a_0x^6 + ...\\
y =& a_0\left[1 - \frac{n(n-1)}{(2n-1)2}x^2 + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)4.2}x^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5)6.4.2}x^6 + ...\right]
\end{aligned}$$

We call this as Legendre polynomial solution of first kind. It is denoted as $\mathrm{P}_n(x)$. That is to say

$$P_n(x) = a_0\left[1 - \frac{n(n-1)}{(2n-1)2}x^2 + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)4.2}x^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5)6.4.2}x^6 + ...\right] \tag{1.9}$$

Case I: When $m = -(n+1)$

In the similar notion we can have the connection between the co-effs. as well as a solution for the differential equation.

$$a_{k+2} = \frac{(n+k+1)(n+k+2)}{(2n+k+3)(k+2)}a_k \tag{1.10}$$

$$\begin{aligned} k =&0, \qquad a_2 = \frac{(n+1)(n+2)}{(2n+3)2}a_0 \\ k =&2, \qquad a_4 = \frac{(n+3)(n+4)}{(2n+5)4}a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+3)(2n+5)2.4} \end{aligned}$$

$$y = a_0\left[x^{-(n+1)} + \frac{(n+1)(n+3)}{(2n+3)2}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+3)(2n+5)4.2}x^{-(n+5)} + ...\right] \tag{1.11}$$

We call this as Legendre polynomial solution of second kind. It is denoted as $\mathrm{Q}_n(x)$. That is

$$Q_n(x) = a_0\left[x^{-(n+1)} + \frac{(n+1)(n+3)}{(2n+3)2}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+3)(2n+5)4.2}x^{-(n+5)} + ...\right] \tag{1.12}$$

The general solution of a non-negative integer degree Legendre's Differential Equation can hence be expressed as

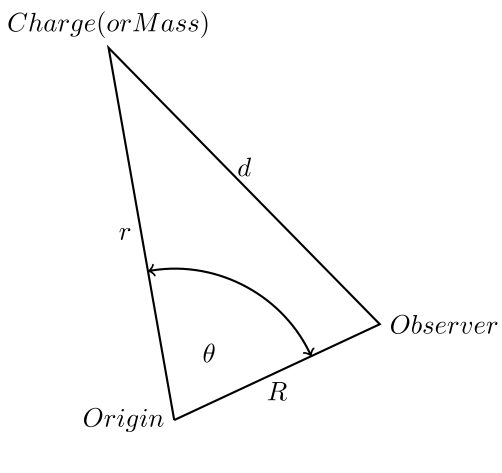
$$y(x) = A_nP_n(x) + B_nQ_n(x) \tag{1.13}$$

However, $\mathrm{Q}_n(x)$ is divergent at $[-1, 1]$. Therefore, the associated coefficient B_n is forced to be zero to obtain a physically meaningful result when there are no sources or sinks at the boundary points $x = \pm 1$. Dear students this required because we need to have a solution that finds some place in physics to be applied for.

1.3 Generating Function (GF) for Legendre Polynomials

A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag. by George Polya

In mathematics, a generating function describes an infinite sequence of numbers (a_n) by treating them like the coefficients of a series expansion. The sum of this infinite series is the generating function. There are various types of generating functions, including ordinary generating functions, exponential generating functions, Lambert series, Bell series, and Dirichlet series etc. But I must say that the particular generating function, if any, that is most useful in a given context will depend upon the nature of the details of the problem being addressed.
I will show now how a completely different analysis, one arising from a well known physical situation, also yields the derivation of Legendre polynomials. This analysis will also allow us to write the **generating function** for Legendre polynomials. Understanding the generating function will allow us to investigate these polynomials more deeply, and allow us to find many useful relationships.First, we start with a well known physical situation as depicted below



This figure shows a particle at the head of vector **r** with respect to a fixed origin. An observer is located at the end of the vector denoted as **R**. Quite often in physics, we want to describe the potential field measured at **R** generated by the particle located at **r**. In this case it does not matter if we are interested in the potential arising from an electric or gravitational field of the particle; since both fields follow inverse square laws, their potentials are expressible in the same mathematical format. We know from basic physics that the potential of a $\frac{1}{r^2}$ field goes as $\frac{1}{r}$, so that the potential of the particle at the observer will go as $\frac{K}{d}$, where K is some constant and d is the magnitude of the vector **d** as shown in the above figure. Now If we wish to express the potential at the observer in terms of the coordinate system in which the origin is at $(0,0,0)$, we need to write **d** in terms of **r**, **R**, and θ. The law of cosines tells us that:

$$\begin{aligned} d^2 =& r^2 + R^2 - 2rRcos\theta \\ d =& \sqrt{r^2 + R^2 - 2rRcos\theta} \\ d =& R\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\frac{r}{R}cos\theta} \end{aligned}$$

If we assume $\frac{r}{R} = z$ and $cos\theta = x$ the last expression gives $d = R\sqrt{1 - 2xz + z^2}$. Following the same footsteps we are now in a position to have our GF in case LDE. And following is the celebrated GF for LDE

$$\boxed{\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n} \tag{1.14}$$

Let us now prove whether the above GF gives us all the different order polynomials or not.

PROOF

All of you know from your Higher secondary 2nd year class how to expand some equation binomially. If you can't remember them now, I would like to suggest you to go through those mathematics books which you have studied during your HS. If you don't to all you can do is just google it in your android mobile phone. Because we will be using lots of these binomial expansions.

$$\frac{1}{(1+y)^m} = 1 - my + \frac{m(m+1)}{2!}y^2 - \frac{m(m+1)(m+2)}{3!}y^3 + \frac{m(m+1)(m+2)(m+3)}{4!}y^4 + ...$$

So we can just have the LHS of equ^n(14) with little algebraic manipulation as $\frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}}$

Thus we actually can compare y with $\left(z^2 - 2xz\right)$ and m with $\frac{1}{2}$

So we can put back these values in the binomial expansion to get the following

$$\frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} = 1 - \frac{1}{2}(z^2 - 2xz) + \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}(z^2-2xz)^2 - \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{3!}(z^2-2xz)^3 + ...$$

Now we will distribute on each of the terms of above series and will make some further simplification by collecting the co-eff. of same powers of z

$$\begin{aligned} \frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} =& 1 - \frac{1}{2}(z^2 - 2xz) + \frac{\frac{1}{2}(\frac{3}{2})}{2!}(z^4 - 4xz^3 + 4x^2z^2) - \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})}{3!}(z^6 - 6xz^5 + 12x^2z^4 - 8x^3z^3) + ... \\ \frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} =& 1 - \frac{1}{2}(z^2 - 2xz) + \frac{3}{2^2.2!}(z^4 - 4xz^3 + 4x^2z^2) - \frac{3.5}{2^3.3!}(z^6 - 6xz^5 + 12x^2z^4 - 8x^3z^3) + ... \\ \frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} =& 1 - \frac{1}{2}z^2 + xz + \frac{3}{8}z^4 - \frac{3}{2}xz^3 + \frac{3}{2}x^2z^2 - \frac{15}{48}z^6 + \frac{15}{8}xz^5 - \frac{15}{4}x^2z^4 + \frac{15}{6}x^3z^3 + ... \\ \frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} =& 1 + xz + (\frac{3}{2}x^2 - \frac{1}{2})z^2 + (\frac{15}{6}x^3 - \frac{3}{2}x)z^3 + (\quad some \quad terms \quad)z^4 ... \\ \frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} =& 1 + xz + (\frac{3}{2}x^2 - \frac{1}{2})z^2 + (\frac{5}{2}x^3 - \frac{3}{2}x)z^3 + (\quad some \quad terms \quad)z^4 + ... \end{aligned}$$

Thus we note that the co-eff. of different powers of z are nothing but the various order polynomial of LDE. Thus

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = (\frac{3}{2}x^2 - \frac{1}{2}), \qquad P_3(x) = (\frac{5}{2}x^3 - \frac{3}{2}x) \quad etc$$

$$\frac{1}{[1+(z^2-2xz)]^{\frac{1}{2}}} = P_0(x)z^0 + P_1(x)z + P_2(x)z^2 + P_3(x)z^3 + ... \tag{1.15}$$

Thus consising we get

$$\boxed{\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n} \tag{1.16}$$

1.4 The Rodriques Formula

The first property that the Legendre polynomials have is the Rodrigues formula. From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function. The Rodriques Formula is given by

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n \in N_0} \tag{1.17}$$

Let us now prove it. The method of proof is simple and very naive one. All we are going to do is to differentiate the expression and in addition to that some ordinary algebra.

PROOF

Let us assume $\nu = (x^2-1)^n$ and then differentiate this with respect to x. That is

$$\begin{aligned}\nu =& (x^2-1)^n \\ \frac{d\nu}{dx} =& n(x^2-1)^{n-1}2x\end{aligned}$$

Now Multiplying both side in the above expression with (x^2-1) we get

$$\begin{aligned}(x^2-1)\frac{d\nu}{dx} =& n(x^2-1)^n 2x \\ (x^2-1)\frac{d\nu}{dx} =& 2nx\nu \qquad\qquad \nu = (x^2-1)^n\end{aligned}$$

Here we need to find a generalised expression to get some idea about the co-eff. those will be coming after each successive differentiation so that we can finally generalise those co-eff. for nth or$(n+1)$th times differentiation. Keeping in view of this let us now differntiate the above expression

$$\begin{aligned}(x^2-1)\frac{d^2\nu}{dx^2} + 2x\frac{d\nu}{dx} =& 2nx\frac{d\nu}{dx} + 2n\nu \qquad\qquad (\frac{d}{dx} \to once) \\ (x^2-1)\frac{d^3\nu}{dx^3} + 2x\frac{d^2\nu}{dx^2} + 2x\frac{d^2\nu}{dx^2} + 2\frac{d\nu}{dx} =& 2nx\frac{d^2\nu}{dx^2} + 2n\frac{d\nu}{dx} + 2n\frac{d\nu}{dx} \\ (x^2-1)\frac{d^3\nu}{dx^3} + 4x\frac{d^2\nu}{dx^2} + 2\frac{d\nu}{dx} =& 2nx\frac{d^2\nu}{dx^2} + 4n\frac{d\nu}{dx} \qquad\qquad (\frac{d}{dx} \to twice) \\ (x^2-1)\frac{d^4\nu}{dx^4} + 2x\frac{d^3\nu}{dx^3} + 4x\frac{d^3\nu}{dx^3} + 4\frac{d^2\nu}{dx^2} + 2\frac{d^2\nu}{dx^2} =& 2nx\frac{d^3\nu}{dx^3} + 2n\frac{d^2\nu}{dx^2} + 4n\frac{d^2\nu}{dx^2} \\ (x^2-1)\frac{d^4\nu}{dx^4} + 6x\frac{d^3\nu}{dx^3} + 6\frac{d^2\nu}{dx^2} =& 2nx\frac{d^3\nu}{dx^3} + 6n\frac{d^2\nu}{dx^2} \qquad\qquad (\frac{d}{dx} \to thrice)\end{aligned}$$

Now a slight transformation and reorientation in the last expression will yeild

$$\begin{aligned}(x^2-1)\frac{d^4\nu}{dx^4} + 6x\frac{d^3\nu}{dx^3} - 2nx\frac{d^3\nu}{dx^3} + 6\frac{d^2\nu}{dx^2} - 6n\frac{d^2\nu}{dx^2} =& 0 \\ (x^2-1)\frac{d^4\nu}{dx^4} + 2x(3-n)\frac{d^3\nu}{dx^3} + 2(3-3n)\frac{d^2\nu}{dx^2} =& 0 \qquad\qquad (\frac{d}{dx} \to thrice)\end{aligned}$$

Thus we see that if we differentiate 3 times we get the highest order as $\frac{d^4\nu}{dx^4}$ ie $\frac{d^{3+1}\nu}{dx^{3+1}}$ and some co-eff. associated with each order of differentiation. Thus generalising if we differentiate $(n+1)$th times we get the highest order as $\frac{d^{(n+1)+1}\nu}{dx^{(n+1)+1}}$.

Similarly the co-eff. can also be generalised for such number of differentiation. Hence

$$\begin{aligned}
(x^2-1)\frac{d^{n+2}\nu}{dx^{n+2}} + 2x({}^{n+1}C_1 - n)\frac{d^{n+1}\nu}{dx^{n+1}} + 2({}^{n+1}C_2 - {}^{n+1}C_1 n)\frac{d^n\nu}{dx^n} &=0\\
(x^2-1)\frac{d^{n+2}\nu}{dx^{n+2}} + 2x(n+1-n)\frac{d^{n+1}\nu}{dx^{n+1}} + 2\left[\frac{n(n+1)}{2} - n(n+1)\right]\frac{d^n\nu}{dx^n} &=0\\
(x^2-1)\frac{d^{n+2}\nu}{dx^{n+2}} + 2x\frac{d^{n+1}\nu}{dx^{n+1}} - n(n+1)\frac{d^n\nu}{dx^n} &=0
\end{aligned}$$

If we now let $y = \frac{d^n\nu}{dx^n}$ the last expression becomes

$$\begin{aligned}
(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y &=0\\
(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y &=0
\end{aligned}$$

This shows that $y = \frac{d^n\nu}{dx^n}$ is a solution of LDE. Thus we can generalise as

$$\begin{aligned}
P_n &=C\frac{d^n\nu}{dx^n}\\
P_n &=\frac{1}{2^n n!}\frac{d^n\nu}{dx^n}\\
P_n &=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n
\end{aligned}$$

That's how it can be proved. But sorry students do not ask me where the hell $C = \frac{1}{2^n n!}$ quantity comes. I request you kindly to remember this just like you did in many ocassions in your life. Thus the Rodrique's formula can be used for finding out the expression for various order polynomials. Here are just a few.

Note that the order will indicate the starting power of the polynomial, ie if $n = 4$ the power of x will start with x^4.

Table 1.1: Different order Legendre Polynomials.

order	The polynomial	Expression	order	The polynomial	Expression
n= 0	$P_0(x)$	1	n= 3	$P_3(x)$	$\frac{1}{2}(5x^3-3x)$
n= 1	$P_1(x)$	x	n= 4	$P_4(x)$	$\frac{1}{8}(35x^4-30x^2+3)$
n= 2	$P_2(x)$	$\frac{1}{2}(3x^2-1)$	n= 5	$P_5(x)$	$\frac{1}{8}(63x^5-70x^3+15x)$

1.5 Orthogonality Relationships for Legendre Polynomials

This property turns out to be of vital importance in quantum mechanics, where the polynomials form the basis of the associated Legendre functions, which in turn form part of the solution of the three-dimensional Schrdinger equation. Basis is something upon which an entire foundation can be done.

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \delta_{mn}\frac{2}{2n+1} \qquad \begin{aligned}\delta_{mn} &= 0 \quad n \neq m\\ \delta_{mn} &= 1 \quad n = m\end{aligned}$$

Lets now prove the above. It's very important form examination point of view. We will prove it part by part. First we will show how it can be shown to zero then the other one ie $\frac{2}{2n+1}$.

First for $\delta_{mn} = 0$ ie when $n \neq m$

By now we have come to know that $y = P_n(x)$ is a solution for LDE.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Now I will multiply the above expression by z keeping in mind that $z = P_m(x)$ is also a solution of LDE

$$(1-x^2)z\frac{d^2y}{dx^2} - 2xz\frac{dy}{dx} + n(n+1)yz = 0 \tag{1.18}$$

Similarly for $z = P_m(x)$ the LDE is

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0$$

Now I multiply the above expression with y to get the following

$$(1-x^2)y\frac{d^2z}{dx^2} - 2xy\frac{dz}{dx} + m(m+1)yz = 0 \tag{1.19}$$

Now substracting equ(19) from equ(18) we get

$$(1-x^2)\left[z\frac{d^2y}{dx^2} - y\frac{d^2z}{dx^2}\right] - 2x\left[z\frac{dy}{dx} - y\frac{dz}{dx}\right] + [n(n+1) - m(m+1)]yz = 0$$

Now we will do slight algebraic manipulation in the above so that some tricky algebra can be done as the following

$$\begin{aligned}(1-x^2)\left[z\frac{d^2y}{dx^2} + \frac{dz}{dx}\frac{dy}{dx} - \frac{dz}{dx}\frac{dy}{dx} - y\frac{d^2z}{dx^2}\right] - 2x\left[z\frac{dy}{dx} - y\frac{dz}{dx}\right] + [n^2+n-m^2-m)]yz &= 0\\ \frac{d}{dx}\left[(1-x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right] + [(n-m)(n+m+1)]yz &= 0\end{aligned}$$

Now integrating the above expression with respect to x from -1 to 1 since within $[-1,1]$ the LDE converges for any value of x as x has regular singular points

$$\begin{aligned}\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]dx + (n-m)(n+m+1)\int_{-1}^{1} yzdx &= 0\\ \left[(1-x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_{-1}^{1} + (n-m)(n+m+1)\int_{-1}^{1} yzdx &= 0\end{aligned}$$

Here you can do a simple calculation to show the first term in the left hand side is going to be zero within the stipulated limits. Thus we are left with the following

$$0 + (n-m)(n+m+1)\int_{-1}^{1} yzdx = 0$$

Here we have two choices. Either $(n-m)(n+m+1) = 0$ or $\int_{-1}^{1} yzdx = 0$. But since $n \neq m$ therefore the integral part has to be zero. That means

$$\boxed{\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad n \neq m} \tag{1.20}$$

<u>Now for $\delta_{mn} = 1$ ie when $n = m$</u>

In that case, the integration by parts technique wont work, since we cant count on the final integral being zero. Therfore to prove this one we will start with the generating function of LDE. There are other methods to prove that. But I find this one is easy and some of the principles of mathematics are previously known to you like binomial expansion. Aha here again binomial comes. That's why I was more emphasising on it in previous sections. This is not gonna leave you.
Ok we our generating function as

$$(1-2xz+z^2)^{-\frac{1}{2}} = \sum z^n P_n(x)$$

Now I will square it up on both sides. And I will get the following

$$(1-2xz+z^2)^{-1} = \sum z^{2n}P_n^2(x) + 2\sum z^{m+n}P_n(x)P_m(x) \tag{1.21}$$

Not convinced with the 2nd term in the RHS. Ok, so let me put it in the following. See your summation runs from $n = 0$ to $n = \infty$. So it covers all values of n. Let us take only two values of n as if the summation runs from $n = 0$ to $n = 1$. Then $\sum z^n P_n(x)$ will yeild the following

$$\sum_{n=0}^{1} z^n P_n(x) = z^0 P_0(x) + z^1 P_1(x)$$

Now if I square this up what will I get the following

$$
\begin{aligned}
\left[\sum_{n=0}^{1} z^n P_n(x)\right]^2 &= \left[z^0 P_0(x) + z^1 P_1(x)\right]^2 \\
&=(z^0)^2 P_0^2(x) + (z^1)^2 P_1^2(x) + 2z^0 z^1 P_0(x) P_1(x) \\
&=\sum_{n=0}^{1} z^{2n} P_n^2(x) + 2\sum z^{m+n} P_n(x) P_m(x)
\end{aligned}
$$

That the sum of the first two terms will give $\sum_{n=0}^{1} z^{2n} P_n^2(x)$ and the third term will give $2\sum z^{m+n} P_n(x)P_m(x)$. Right! Not satisfied! Ok Let us do that for three terms. ie summation runs from $n = 0$ to $n = 2$

$$
\begin{aligned}
\left[\sum_{n=0}^{2} z^n P_n(x)\right]^2 &= \left[z^0 P_0(x) + z^1 P_1(x) + z^2 P_2(x)\right]^2 \\
&=(z^0)^2 P_0^2(x) + (z^1)^2 P_1^2(x) + (z^2)^2 P_2^2(x) + 2z^0 z^1 P_0(x) P_1(x) + 2z^0 z^2 P_0(x) P_2(x) + 2z^1 z^2 P_1(x) P_2(x) \\
&=\sum_{n=0}^{2} z^{2n} P_n^2(x) + 2\sum z^{m+n} P_n(x) P_m(x)
\end{aligned}
$$

Now you see that the sum of the first three terms will give $\sum_{n=0}^{2} z^{2n} P_n^2(x)$ and the last three term will give $2\sum z^{m+n} P_n(x)P_m(x)$. If you are convinced let us now move forward. Now we will be integrating the equ^n(21) with respect to x over the limits -1 to 1.

$$
\begin{aligned}
\int_{-1}^{1} \sum z^{2n} P_n^2(x) dx + \int_{-1}^{1} 2\sum z^{m+n} P_n(x) P_m(x) dx &= \int_{-1}^{1} (1 - 2xz + z^2)^{-1} dx \\
\sum z^{2n} \int_{-1}^{1} P_n^2(x) dx + 2\sum z^{m+n} \int_{-1}^{1} P_n(x) P_m(x) dx &= \int_{-1}^{1} (1 - 2xz + z^2)^{-1} dx \\
\int_{-1}^{1} \sum z^{2n} P_n^2(x) dx + 0 &= \int_{-1}^{1} \frac{1}{1 - 2xz + z^2} dx \qquad \int_{-1}^{1} P_n(x) P_m(x) dx = 0 \quad n \neq m \\
&= \int_{-1}^{1} -\frac{1}{2z} \left(\frac{-2z}{1 - 2xz + z^2}\right) dx
\end{aligned}
$$

So here in RHS we have something like *diffrentiation of the numerator is equal to the denominator the value of the intergral will be log of the denominator.* Therefore

$$
\begin{aligned}
\int_{-1}^{1} \sum z^{2n} P_n^2(x) dx &= -\frac{1}{2z} log\left[1 - 2xz + z^2\right]_{-1}^{1} \\
&= -\frac{1}{2z} log\left[\frac{1 - 2z + z^2}{1 + 2z + z^2}\right] \\
&= -\frac{1}{2z} log\left[\frac{(1 - z)^2}{(1 + z)^2}\right] \\
&= -\frac{1}{z} log\left[\frac{(1 - z)}{(1 + z)}\right]
\end{aligned}
$$

$$\begin{aligned}
\int_{-1}^{1}\sum z^{2n}P_n^2(x)dx =&\frac{1}{z}log\left[\frac{(1+z)}{(1-z)}\right]\\
=&\frac{1}{z}\left[log(1+z)-log(1-z)\right]\\
=&\frac{1}{z}\left[(z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}-\ldots)-(-z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\frac{z^5}{5}-\ldots)\right]\\
=&\frac{1}{z}\left[z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots+z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\ldots\right]\\
=&\frac{1}{z}\left[2z+\frac{2z^3}{3}+\frac{2z^5}{5}+\frac{2z^7}{7}\ldots\right]\\
=&2\left[1+\frac{z^2}{3}+\frac{z^4}{5}+\frac{z^6}{7}\ldots)\right]\\
=&2\left[1+\frac{z^2}{3}+\frac{z^4}{5}+\frac{z^6}{7}\ldots+\frac{z^{2n}}{2n+1}\right]\\
=&\sum\frac{2}{2n+1}z^{2n}
\end{aligned}$$

Now if you equate the co-eff. of z^{2n} you will get

$$\boxed{\int_{-1}^{1}P_n^2(x)dx=\frac{2}{2n+1}\qquad n=m} \tag{1.22}$$

1.6 Recurrence Relationships for Legendre Polynomials

In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. Thus a recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s).The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. To generate sequence basd on a recurrence relation, one must start with some initial values. If we denote the nth term in the sequence by x_n, such a recurrence relation is of the form $x_{n+1}=f(x_n)$. Sometimes it also refers to as **difference equation** to indicate some specific type of recurrence relation. In case Legendre differential equation we have six number of such equations. One can have even more. One just have play around the equations. These are in terms of different order polynomials or in terms of derivative of polynomial or a hybrid of both these. Let's investigate them.

First reccurence relation

Step I: Take the generating function (GF)

$$(1-2xz+z^2)^{-\frac{1}{2}}=\sum z^nP_n(x)$$

Step II: Differentiate the above expression w. r. to. z

$$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2x+2z)=\sum nz^{n-1}P_n(x)$$

Step III: Multiply both sides with$(1-2xz+z^2)$

$$(1-2xz+z^2)^{-\frac{1}{2}}(x-z)=(1-2xz+z^2)\sum nz^{n-1}P_n(x) \tag{1.23}$$

$$\sum z^nP_n(x)(x-z)=(1-2xz+z^2)\sum nz^{n-1}P_n(x) \tag{1.24}$$

Step IV: Distribute the terms in either side

$$\sum z^nxP_n(x)-\sum z^{n+1}P_n(x)=\sum nz^{n-1}P_n(x)-2n\sum z^nxP_n(x)+\sum nz^{n+1}P_n(x)$$

Step IV: Now equate the co-efficient of z^{n-1}

As this is a summation runs from 0 to ∞ therefore the series contains all the terms including the terms with different powers of z along with the polynomials. From there we have to pick the coefficients of z^{n-1}. Also I will be leaving (x) (ie the so called function of x as it is tiresome). Thus in a straight forward manner if we replace all the 'n's by either 'n-1's or 'n-2's except in the terms where there are already 'n-1' present, then we get

$$\sum z^{(n-1)} x P_{(n-1)} - \sum z^{(n-2)+1} P_{(n-2)} = \sum n z^{n-1} P_n - 2\sum (n-1) z^{(n-1)} x P_{(n-1)} + \sum (n-2) z^{(n-2)+1} P_{(n-2)}$$

$$xP_{n-1} - P_{n-2} = nP_n - 2(n-1)xP_{n-1} + (n-2)P_{n-2}$$

$$xP_{n-1} - P_{n-2} = nP_n - 2nxP_{n-1} + 2xP_{n-1} + nP_{n-2} - 2P_{n-2}$$

$$nP_n - 2nxP_{n-1} + xP_{n-1} + nP_{n-2} - P_{n-2} = 0$$

$$nP_n = (2n-1)xP_{n-1} + (n-1)P_{n-2}$$

Thus the first recurrence relationship is $\boxed{nP_n = (2n-1)xP_{n-1} + (n-1)P_{n-2}}$

Second reccurence relation

Step I: Take the generating function (GF)

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum z^n P_n(x)$$

Step II: Differentiate the above expression w. r. to. z

$$-\frac{1}{2}(1 - 2xz + z^2)^{-\frac{3}{2}}(-2x + 2z) = \sum n z^{n-1} P_n(x) \tag{1.25}$$

Step II: Differentiate the GF w. r. to. x

$$-\frac{1}{2}(1 - 2xz + z^2)^{-\frac{3}{2}}(-2z) = \sum z^n P_n'(x) \tag{1.26}$$

Step III: Divide equ^n (25) by equ^n (26)

$$\frac{x - z}{z} = \frac{\sum n z^{n-1} P_n(x)}{\sum z^n P_n'(x)}$$

$$(x - z)\sum z^n P_n'(x) = z\sum n z^{n-1} P_n(x)$$

$$x\sum z^n P_n'(x) - \sum z^{n+1} P_n'(x) = \sum n z^n P_n(x)$$

Step IV: Now equate the co-efficient of z^n in the last expression just like you did earlier in RR-I

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \tag{1.27}$$

Thus the second recurrence relationship is $\boxed{xP_n'(x) - P_{n-1}'(x) = nP_n(x)}$

Third reccurence relation

Step I: Take the recurrence relationship I (RR-I)

$$nP_n = (2n-1)xP_{n-1} + (n-1)P_{n-2} \tag{1.28}$$

Step II: Differentiate the above expression w. r. to. x

$$nP_n' = (2n-1)P_{n-1} + (2n-1)xP_{n-1}' - (n-1)P_{n-2}'$$

$$nP_n' = (2n-1)P_{n-1} + nxP_{n-1}' + (n-1)xP_{n-1}' - (n-1)P_{n-2}'$$

$$nP_n' - nxP_{n-1}' = (2n-1)P_{n-1} + (n-1)xP_{n-1}' - (n-1)P_{n-2}'$$

$$n(P_n' - xP_{n-1}') = (n-1)(xP_{n-1}' - P_{n-2}') + (2n-1)P_{n-1}$$

Step III: Use RR-II in the first term of the RHS. As it is an identity equation with one order less with RR-II. Thus

$$xP_{n-1}' - P_{n-2}' = (n-1)P_{n-1}$$

Replacing the above term in last expression we get

$$\begin{aligned} n(P'_n - xP'_{n-1}) &= (n-1)(n-1)P_{n-1} + (2n-1)P_{n-1} \\ n(P'_n - xP'_{n-1}) &= [(n-1)^2 + (2n-1)]P_{n-1} \\ n(P'_n - xP'_{n-1}) &= (n^2 - 2n + 1 + 2n - 1)P_{n-1} \\ n(P'_n - xP'_{n-1}) &= n^2 P_{n-1} \\ P'_n - xP'_{n-1} &= nP_{n-1} \end{aligned}$$

Thus the third recurrence relationship is $\boxed{P'_n - xP'_{n-1} = nP_{n-1}}$

NOTE: Thus in order to proof RR-III you have to first proof RR-I. Of course that will depend upon the marks given in the question paper. Generally it will come as 4 marks question. Anything more than that you are to proof RR-I.

Fourth reccurence relation

Step I: Take the recurrence relationship I (RR-I)

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$$

Step II: Replace n by n+1 in the above expression

$$\begin{aligned} (n+1)P_{n+1} &= [2(n+1)-1)xP_{(n+1)-1} - [(n+1)-1]P_{(n+1)-2} \\ (n+1)P_{n+1} &= (2n+1)xP_n - nP_{n-1} \end{aligned}$$

Step III: Differentiate the above expression w. r. to. x

$$(n+1)P'_{n+1} = (2n+1)P_n + (2n+1)xP'_n - nP'_{n-1}$$

Step IV: Use RR-II for the term xP'_n

$$\begin{aligned} (n+1)P'_{n+1} &= (2n+1)P_n + (2n+1)(P'_{n-1} + nP_n) - nP'_{n-1} \\ (n+1)P'_{n+1} &= (2n+1)P_n + (2n+1)P'_{n-1} + (2n+1)nP_n - nP'_{n-1} \\ (n+1)P'_{n+1} &= (2n+1)P_n + 2nP'_{n-1} + P'_{n-1} + (2n+1)nP_n - nP'_{n-1} \\ (n+1)P'_{n+1} &= (2n+1)P_n + nP'_{n-1} + P'_{n-1} + (2n+1)nP_n \\ (n+1)P'_{n+1} &= (2n+1)P_n + (n+1)P'_{n-1} + (2n+1)nP_n \\ (n+1)P'_{n+1} - (n+1)P'_{n-1} &= (2n+1)P_n + 2nP'_{n-1} + (2n+1)nP_n) \\ (n+1)P'_{n+1} - (n+1)P'_{n-1} &= (2n+1)(n+1)P_n + 2n(P'_{n-1}) \\ P'_{n+1} - P'_{n-1} &= (2n+1)P_n \end{aligned}$$

Thus the fourth recurrence relationship is $\boxed{P'_{n+1} - P'_{n-1} = (2n+1)P_n}$

NOTE: Thus in order to proof RR-IV you have to first proof RR-I. Then of course RR-II as we have used these. But then it will again depend upon the marks given in the question paper.

Fifth reccurence relation

Step I: Take the recurrence relationship III (RR-III)

$$P'_n - xP'_{n-1} = nP_{n-1}$$

Step II: Take the recurrence relationship II (RR-II)

$$xP'_n - P'_{n-1} = nP_n \tag{1.29}$$

Step III: Multiply the RR-II with x

$$x^2P'_n - xP'_{n-1} = nxP_n \tag{1.30}$$

Step IV: substract equ^n(29) from equ^n(30)

$$(x^2-1)P'_n = n(xP_n - P_{n-1}) \tag{1.31}$$

Thus the fifth recurrence relationship is $\boxed{(x^2-1)P'_n = n(xP_n - P_{n-1})}$

NOTE: Thus in order to proof RR-V you have to first prove RR-III. Then for RR-III you are to prove RR-I. Then of course RR-II has to be proved again as we have used these. But then it will again depend upon the marks given in the question paper.

Sixth reccurence relation

Step I: Take the recurrence relationship I (RR-I)

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$$

Step II: Replace n by n+1 in the above expression

$$\begin{aligned}
(n+1)P_{n+1} &= [2(n+1)-1)xP_{(n+1)-1} - [(n+1)-1]P_{(n+1)-2} \\
(n+1)P_{n+1} &= (2n+1)xP_n - nP_{n-1} \\
(n+1)P_{n+1} &= (n+1)xP_n + nxP_n - nP_{n-1} \\
(n+1)P_{n+1} - (n+1)xP_n &= nxP_n - nP_{n-1} \\
(n+1)[P_{n+1} - xP_n] &= nxP_n - nP_{n-1}
\end{aligned}$$

Step II: Use recurrence relation V (RR-V) in the RHS of the above expression

$$(n+1)[P_{n+1} - xP_n] = (x^2-1)P'_n$$

Thus the sixth recurrence relationship is $\boxed{(x^2-1)P'_n = (n+1)[P_{n+1} - xP_n]}$

NOTE: Thus in order to proof RR-VI you have to first prove RR-I. Then RR-V you are to prove RR-I. Sorry students I could not do better than this. It's all about manuvreing your stipulated time in the exam hall.

Chapter 2

Hermite Differential Equation (HDE)

2.1 Introduction

Whaoo! Another standard differential equation. Hermite polynomials were defined by Laplace (1810) though in scarcely recognizable form, and studied in detail by Chebyshev (1859). Chebyshev's work was overlooked and they were named later after Charles Hermite who wrote on the polynomials in 1864 describing them as new.
The HDE is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \tag{2.1}$$

2.2 Properties of HDE:

1) Well it has regular singular points for every value of x. You can see it by yourself. By now you know how to do that.
2) Thus it has a range of points $[-\infty, \infty]$ including 0 as regular singular points.
3) In the Hermites equation, for a special choices of n give rise to the Hermite polynomials just like we have in the case of Legendre.
4) The solution for HDE can be expanded both as Frobenius method or Taylor's method (It's because there is no specific value of x as regular singular points .
5) HDE finds place in numerical analysis as **Gaussian quadrature**; in probability, such as the **Edgeworth series**; in physics, where they give rise to the **eigenstates of the quantum harmonic oscillator**

All right. Having ourself gearing up let's now move to get solution for HDE. As we have done the Frobenius method earlier therefore assume the series solution of HDE as

$$\begin{aligned} y =& \sum_{k=0}^{\infty} a_k x^{m+k} \\ y' =& (m+k)\sum_{k=0}^{\infty} a_k x^{m+k-1} \\ y'' =& (m+k)(m+k-1)\sum_{k=0}^{\infty} a_k x^{m+k-2} \end{aligned}$$

Next time onwards I am not gonna write the summation sign as well as its limits. Because it's really tiresome. But you are requested to keep that in mind even if I am not writing that it is always there. Let us now back substitute the above equations in the original HDE and do some little bit of normal algebra.

$$\begin{aligned} (m+k)(m+k-1)a_k x^{m+k-2} - 2x(m+k)a_k x^{m+k-1} + 2na_k x^{m+k} =& 0 \\ (m+k)(m+k-1)a_k x^{m+k-2} - 2a_k(m+k)x^{m+k} + 2na_k x^{m+k} =& 0 \\ (m-k)(m+k-1)a_k x^{m+k-2} - a_k[2(m+k) - 2n]x^{m+k} =& 0 \end{aligned}$$

The last equation is an identity and to get the indicial equation we equate the co-efficient of lowest power of x to 0 and in there we will also put $k = 0$. And it's not very tough to identify that. Yes, it's x^{m+k-2}.

$$a_0[(m-0)(m+0-1)] = 0 \tag{2.2}$$

Since $a_0 \neq 0$ therefore

$$m(m-1) = 0$$

Thus the roots of the indicial equations are either $m = 0$ or $m = 1$
But dear students please note one thing here. Even though we are expanding the solution as a series solution but we are not going to take the final solution as we use to do in case Frobenius method. ie roots are distinct and differ by an integer. It's because there is no specific value of x as a regular singular point. Lets us now find the connection between the various order co-efficients. Since again we working according to Frobenius therfore we will be raising the all lower powers of x to the highest powers x. And that can easily be done by replacing all the "k"s by "$k+2$". Thus the indicial equation will yield

$$(m+k+2)(m+k+1)a_{k+2} - a_k[2(m+k) - 2n] = 0$$

Thus

$$a_{k+2} = \frac{2(m+k) - 2n}{(m+k+2)(m+k+1)} a_k \tag{2.3}$$

Case I: When $m = 0$
We can simplify denominator of the last expression putting $m = n$ down to the following

$$\begin{aligned} a_{k+2} =& \frac{2(0+k) - 2n}{(0+k+2)(0+k+1)} a_k \\ a_{k+2} =& \frac{2k - 2n}{(k+2)(k+1)} a_k \end{aligned}$$

now if we put $k = 0$ then we get the connection between a_2 and a_0. ie

$$k = 0, \qquad a_2 = -\frac{2n}{2.1} a_0 \tag{2.4}$$

we now put $k = 2$ to get the relationship between a_4 and a_2. But then a_2 is connected to a_0. That means a_4 can be related to a_0.

$$k = 2, \quad a_4 = \frac{4-2n}{4.3} a_2 = \left[\frac{4-2n}{4.3}\right]\left[-\frac{2n}{2.1} a_0\right] a_0 = -\frac{(4-2n)2n}{4.3.2.1} a_0 = -\frac{(4-2n)2n}{4!} a_0 \tag{2.5}$$

In the similar notion

$$k = 4, \quad a_6 = \frac{(6-2n)(4-2n)2n}{6.5.4.3.2.1} a_0 = -\frac{(8-2n)(4-2n)2n}{6!} a_0 \tag{2.6}$$

Remember that here in HDE there is no specific value of x as a regular singular point. And because of that don't take it to your surprise, we can't he put $a_1 = 0$ by simply letting $k = -1$ and demanding that since -ve co-eff. doesn't exsists unlike in other cases those we have done earlier. Therefore we also need to get the odd co-eff. And our base co-eff. will be a_1. That means we will establish the connection between a_1 and a_3 then a_3 and a_5 and so on. Thus if we put $k = 1$ then we get the connection between a_3 and a_1. Thus from $a_{k+2} = \frac{2k-2n}{(k+2)(k+1)} a_k$ we get

$$k = 1, \qquad a_3 = \frac{2-2n}{3.2} a_1 \tag{2.7}$$

we now put $k = 3$ to get the relationship between a_5 and a_3. But then a_3 is connected to a_1. That means a_5 can be related to a_1.

$$k = 3, \quad a_5 = \frac{(6-2n)}{5.4} a_1 = \left[\frac{(6-2n)}{5.4}\right]\left[\frac{2-2n}{3.2}\right] a_1 = \frac{(6-2n)(2-2n)}{5.4.3.2} a_1 = \frac{(6-2n)(2-2n)}{5!} a_1 \tag{2.8}$$

In the similar notion

$$k = 5, \quad a_7 = \frac{(10-2n)}{7.6} a_5 = \left[\frac{(10-2n)}{7.6}\right]\left[\frac{(6-2n)(2-2n)}{5.4.3.2}\right] a_1 = \frac{(10-2n)(6-2n)(2-2n)}{7!} a_1 \tag{2.9}$$

And so on. Thus we can expand the solution as for $m = 0$

$$y = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \ldots\ldots$$

$$y = a_0 + a_1 x - \frac{2n}{2!} a_0 x^2 + \frac{2-2n}{3.2} a_1 x^3 - \frac{(4-2n)2n}{4!} a_0 x^4 + \frac{(6-2n)(2-2n)}{5!} a_1 x^5 - \frac{(8-2n)(4-2n)2n}{6!} a_0 x^6 + \frac{(10-2n)(6-2n)(2-2n)}{7!} a_1 x^7 - \ldots\ldots$$

$$y = a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - \frac{2^3 n(n-2)(n-4)}{6!} x^6 + \ldots\ldots + (-2)^k \frac{n(n-2)(n-4)\ldots(n-2k+2)}{(2k)!} x^{2k}\right] + a_1 \left[x + \frac{2-2n}{3.2} x^3 + \frac{(6-2n)(2-2n)}{5!} x^5 + \frac{(10-2n)(6-2n)(2-2n)}{7!} a_1 x^7 + \ldots\ldots\right]$$

$$y = a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - \frac{2^3 n(n-2)(n-4)}{6!} x^6 + \ldots\ldots + (-2)^k \frac{n(n-2)(n-4)\ldots(n-2k+2)}{(2k)!} x^{2k}\right] + a_1 x \left[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2(n-3)(n-1)}{5!} x^4 + \frac{2^2(n-5)(n-3)(n-1)}{7!} a_1 x^6 + \ldots + (-2)^k \frac{(n-1)(n-3)\ldots(n-2k+1)}{(2k+1)!} x^{2k}\right]$$

For n even, the series multiplying a_0 terminates so it is the appropriate result for H_n(x). a_1 is necessarily zero. Similarly for n odd, the series multiplying a_1 terminates so it is the appropriate result for H_n(x). a_0 is necessarily zero. We call this as Hermite polynomial solution. It is denoted as $H_n(x)$. That is to say

$$H_n(x) = a_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - \frac{2^3 n(n-2)(n-4)}{6!} x^6 + \ldots\ldots + (-2)^k \frac{n(n-2)(n-4)\ldots(n-2k+2)}{(2k)!} x^{2k}\right] + a_1 x \left[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2(n-3)(n-1)}{5!} x^4 + \ldots + (-2)^k \frac{(n-1)(n-3)\ldots(n-2k+1)}{(2k+1)!} x^{2k}\right]$$

Case II: When $m = 1$

In the similar notion from equ^n(3)we can have the connection between the co-effs. as well as a solution for the differential equation.

$$a_{k+2} = \frac{2(k+1-n)}{(k+3)(k+2)} a_k \tag{2.10}$$

$$\begin{aligned}
k =& 0, \quad a_2 = \frac{2(1-n)}{3.2} a_0 \\
k =& 1, \quad a_3 = \frac{2(2-n)}{4.3} a_1 = \frac{2.2(2-n)}{4!} a_1 \\
k =& 2, \quad a_4 = \frac{2(3-n)}{5.4} a_2 = \frac{2(3-n)}{5.4} \frac{2(1-n)}{3.2} a_0 = \frac{2^2(1-n)(3-n)}{5.4.3.2} a_0 = \frac{2^2(1-n)(3-n)}{5!} a_0 \\
k =& 3, \quad a_5 = \frac{2(4-n)}{6.5} a_3 = \frac{2(4-n)}{6.5} \frac{2(2-n)}{4.3} a_1 = \frac{2.2^2(4-n)(2-n)}{6!} a_1 \\
k =& 4, \quad a_6 = \frac{2(5-n)}{7.6} a_4 = \frac{2^3(5-n)(3-n)(1-n)}{7!} a_0 \\
k =& 5, \quad a_7 = \frac{2(6-n)}{8.7} a_5 = \frac{2^3(6-n)(4-n)(2-n)}{8!} a_1
\end{aligned}$$

Thus now we can write the solution as

$$
\begin{aligned}
y =& a_0x^{0+1} + a_1x^{1+1} + a_2x^{2+1} + a_3x^{3+1} + a_4x^{4+1} + a_5x^{5+1} + a_6x^{6+1} + a_7x^{7+1} \ldots\ldots \\
y =& a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6 + a_6x^7 + a_7x^8 \ldots\ldots \\
y =& x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \ldots\ldots) \\
y =& xa_0\left[1 + \frac{2(1-n)}{3.2}x^2 + \frac{2^2(3-n)(1-n)}{5!}x^4 + \ldots\ldots + (2)^k\frac{(1-n)(3-n)(5-n)\ldots(-1+2k-n)}{(2k+1)!}x^{2k}\right] + \\
& a_1x\left[x + \frac{2^2(2-n)}{4!}x^3 + \frac{2^3(4-n)(2-n)}{6!}x^5 + \ldots + (2)^k\frac{(2-n)(4-n)(6-n)\ldots(-2+2k-n)}{(2k)!}x^{2k+1}\right]
\end{aligned}
$$

$$
\begin{aligned}
y =& xa_0\left[1 - \frac{2(n-1)}{3.2}x^2 + \frac{2^2(n-3)(n-1)}{5!}x^4 + \ldots\ldots + (-2)^k\frac{(n-5)(n-3)(n-1)\ldots(n-2k+1)}{(2k+1)!}x^{2k}\right] + \\
& a_1x^2\left[1 - \frac{2^2(n-2)}{4!}x^2 + \frac{2^3(n-4)(n-2)}{6!}x^4 + \ldots + (-2)^k\frac{(n-6)(n-4)(n-2)\ldots(n-2k+2)}{(2k)!}x^{2k}\right]
\end{aligned}
$$

For n even, the series multiplying a_0 terminates so it is the appropriate result for H_n(x). a_1 is necessarily zero. Similarly for n odd, the series multiplying a_1 terminates so it is the appropriate result for H_n(x). a_0 is necessarily zero. We call this also as Hermite polynomial solution. It is also denoted as $H_n(x)$. That is to say

$$
\begin{aligned}
H_n(x) =& xa_0\left[1 - \frac{2(n-1)}{3.2}x^2 + \frac{2^2(n-3)(n-1)}{5!}x^4 + \ldots\ldots + (-2)^k\frac{(n-5)(n-3)(n-1)\ldots(n-2k+1)}{(2k+1)!}x^{2k}\right] + \\
& a_1x^2\left[1 - \frac{2^2(n-2)}{4!}x^2 + \frac{2^3(n-4)(n-2)}{6!}x^4 + \ldots + (-2)^k\frac{(n-6)(n-4)(n-2)\ldots(n-2k+2)}{(2k)!}x^{2k}\right]
\end{aligned}
$$

Dear students we are very much fortunate that nature is somewhat selective. There is till no system exsists where the term with the co-eff. a_1 finds some place in physics to be applied for. Thus we can safely let $a_1 = 0$. Then you might ask why don't we let it zero just at the beginning while we are trying to find relationship between the co-effs. The answer is because we can't since this has not been done properly as Frobenius method. Remember I told you Frobenius method is valid when $x = 0$ is a regular singular point. Hence the most general solution for HDE is given by

$$
\begin{aligned}
H_n(x) =& a_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2n(n-2)}{4!}x^4 - \frac{2^3n(n-2)(n-4)}{6!}x^6 + \ldots\ldots + (-2)^k\frac{n(n-2)(n-4)\ldots(n-2k+2)}{(2k)!}x^{2k}\right] + \\
& xa_0\left[1 - \frac{2(n-1)}{3!}x^2 + \frac{2^2(n-3)(n-1)}{5!}x^4 + \ldots\ldots + (-2)^k\frac{(n-5)(n-3)(n-1)\ldots(n-2k+1)}{(2k+1)!}x^{2k}\right]
\end{aligned}
$$

Thus we simply boil ourself down to find the value of a_0. But then just like a bolt from the blue I'm gonna give you the value of a_0. And I beg sincerely apology not for telling you where does it comes from and how.

$$
a_0 = (-1)^{\frac{n}{2}}.\frac{n!}{(\frac{n}{2})!} \tag{2.11}
$$

Let's now assume $2k = n$ ie for even values and investigate the term containing x^n from the series with the co-eff. a_0 in the above expression

$$
\begin{aligned}
co-eff.\quad nth\quad term =& (-1)^{\frac{n}{2}}.\frac{n!}{(\frac{n}{2})!}.(-2)^k\frac{n(n-2)(n-4)\ldots(n-2k+2)}{(2k)!}x^{2k} \\
=& (-1)^{\frac{n}{2}}.\frac{n!}{(\frac{n}{2})!}.(-2)^{\frac{n}{2}}.\frac{n(n-2)(n-4)\ldots(n-n+2)}{n!}x^n \\
=& (-1)^{\frac{n}{2}}.\frac{n!}{(\frac{n}{2})!}.(-2)^{\frac{n}{2}}.\frac{2(\frac{n}{2})2\left(\frac{n}{2}-1\right)2\left(\frac{n}{2}-2\right)\ldots2.1}{n!}x^n
\end{aligned}
$$

Remember that here there are $\frac{n}{2}$ no. of terms. As we have taken only the half series. Thus there will be 2 raised to the power of $\frac{n}{2}$

$$
\begin{aligned}
co-eff.\quad nth\quad term =& (-1)^{\frac{n}{2}}.\frac{n!}{(\frac{n}{2})!}.(-2)^{\frac{n}{2}}.2^{\frac{n}{2}}\frac{\frac{n}{2}(\frac{n}{2}-1)(\frac{n}{2}-2)\ldots2.1}{n!}x^n \\
=& \frac{n!}{(\frac{n}{2})!}.2^n\frac{(\frac{n}{2})!}{n!}x^n \\
=& (2x)^n
\end{aligned}
$$

Similarly if we assume $2k + 1 = n$ ie for odd values and investigate the term containing x^n just like above from the series with the co-eff. a_0x in the above expression we will also get it will going it as $(2x)^n$

2.3 The Rodriques Formula

The first property that the Hermite polynomials have is the Rodrigues formula. From the Rodrigues formula, one can show that H$_n$(x) is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function. The Rodriques Formula is given by

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n \in N_0} \tag{2.12}$$

Let us now prove it. The method of proof is simple and very naive one. All we are going to do is to start with doing some ordinary algebraic manipulation. All we need to show that the given formula is indeed the solution for HDE. And the following is how it can be done. It's very simple. Don't get scared. Let's assume

PROOF

$$\begin{aligned} u =& e^{-x^2} \\ u' =& -2xu \\ u' + 2xu =& 0 \\ u'' + 2xu' + 2u =& 0 \qquad (\frac{d}{dx} \to once) \\ u''' + 2xu'' + 2u' + 2u' =& 0 \\ u''' + 2xu'' + 4u' =& 0 \qquad (\frac{d}{dx} \to twice) \end{aligned}$$

Thus we see that if we differentiate 2 times we get the highest order as $\frac{d^3u}{dx^3}$ ie $\frac{d^{2+1}u}{dx^{2+1}}$ and some co-eff. associated with each order of differentiation. Thus generalising if we differentiate $(n+1)$th times we get the highest order as $\frac{d^{(n+1)+1}u}{dx^{(n+1)+1}}$. Similarly the co-eff. can also be generalised for such number of differentiation. Hence

$$u'^{[(n+1)+1]th} + 2xu'^{(n+1)th} + 2(n+1)u'^{nth} = 0$$

If we now let $v = (-1)^n u'^{(nth)}$ the last expression becomes

$$\frac{d^2v}{dx^2} + 2x\frac{dv}{dx} + 2(n+1)v = 0 \tag{2.13}$$

Again if we now let $v = e^{-x^2}y$ then we can generalise as

$$\begin{aligned} v' =& -2xye^{-x^2} + y'e^{-x^2} = (-2xy + y')e^{-x^2} \\ =& -2xv + y'e^{-x^2} \\ v'' =& -2v - 2xv' + y''e^{-x^2} - 2xy'e^{-x^2} \\ =& (y'' - 4xy' + 4x^2y - 2y)e^{-x^2} \end{aligned}$$

Now putting v$''$ and v$'$ in equ^n(15) we get

$$\begin{aligned} (y'' - 4xy' + 4x^2y - 2y)e^{-x^2} + 2x(-2xy + y')e^{-x^2} + 2(n+1)e^{-x^2}y =& 0 \\ (y'' - 4xy' + 4x^2y - 2y) + 2x(-2xy + y') + 2(n+1)y =& 0 \qquad [e^{-x^2} \quad cancels \quad off] \\ y'' - 4xy' + 4x^2y - 2y - 4x^2y + 2xy' + 2(n+1)y =& 0 \\ y'' - 2xy' + 2ny =& 0 \end{aligned}$$

Note that we arrive at the HDE. Thus $y = H_n(x) = ve^{x^2} = (-1)^n u'^{(nth)} e^{x^2} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

Let's now investigate whether we can generate the polynomial from the above equation or not. All we are going to do is just differentiate.

$$\begin{aligned} \frac{d}{dx}e^{-x^2} =& -2xe^{-x^2} \qquad (\frac{d}{dx} \to once) \\ \frac{d^2}{dx^2}e^{-x^2} =& 4x^2e^{-x^2} - 2e^{-x^2} \\ =& (4x^2 - 2)e^{-x^2} \qquad (\frac{d}{dx} \to twice) \\ \frac{d^3}{dx^3}e^{-x^2} =& -8x^3e^{-x^2} + 8xe^{-t^2} + 4xe^{-t^2} \\ =& -(8x^3 - 12x)e^{-x^2} \qquad (\frac{d}{dx} \to thrice) \end{aligned}$$

Thus we see that they can indeed be generated. Hence we have our Rodrique's formula as

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n \in N_0} \tag{2.14}$$

2.4 Generating Function (GF) for Hermite Polynomials

A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag. by George Polya

In mathematics, a generating function describes an infinite sequence of numbers (a_n) by treating them like the coefficients of a series expansion. The sum of this infinite series is the generating function. There are various types of generating functions, including ordinary generating functions, exponential generating functions, Lambert series, Bell series, and Dirichlet series etc. But I must say that the particular generating function, if any, that is most useful in a given context will depend upon the nature of the details of the problem being addressed.

$$\boxed{e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n} \tag{2.15}$$

Let us now prove whether the above GF gives us all the different order polynomials or not.
PROOF

Again go back to your Higher secondary 2nd year class to see how to expand some equation binomially. If you can remember them, that's excellent otherwise go through those mathematics books which you have studied during your HS or eqivalently just google it in your android mobile phone. All I need you to remember is the following

$$e^y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

So we can just have the LHS of equ^n(46) with little algebraic manipulation as $y = 2zx - z^2$ and expand it

$$\begin{aligned} e^{2zx-z^2} =& 1 + (2zx - z^2) + \frac{(2zx - z^2)^2}{2!} + \frac{(2zx - z^2)^3}{3!} + \frac{(2zx - z^2)^4}{4!} + \ldots\ldots \\ =& 1 + 2xz - \frac{2}{2}z^2 + \frac{(2zx)^2 - 2.2zx.z^2 + (z^2)^2}{2!} + \frac{(2zx)^3 - 3.(2zx)^2.z^2 + 3.2zx.(z^2)^2 - (z^2)^3}{3!} + \ldots\ldots \end{aligned}$$

Now we will distribute on each of the terms of above series and will make some further simplification by collecting the co-eff. of same powers of z

$$\begin{aligned} e^{2zx-z^2} =& 1 + 2xz - \frac{2}{2}z^2 + \frac{4z^2x^2}{2} - \frac{4x.z^3}{2} + \frac{z^4}{2} + \frac{8x^3z^3}{6} - \frac{12x^2z^4}{6} + \frac{6xz^3}{6} - \frac{z^6}{6} + \ldots\ldots \\ =& 1 + 2xz + \frac{1}{2!}(4x^2 - 2)z^2 + \frac{1}{3!}(8x^3 - 12x)z^3 + \ldots\ldots \\ =& \frac{H_0(x)}{0!}z^0 + \frac{H_1(x)}{1!}z + \frac{H_2(x)}{2!}z^2 + \frac{H_3(x)}{3!}z^3 + \ldots\ldots \end{aligned}$$

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n \tag{2.16}$$

Thus we note that the co-eff. of different powers of z are nothing but the various order polynomial of HDE. Thus

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = (4x^2 - 2), \qquad H_3(x) = (8x^3 - 12x) \quad etc$$

2.5 Orthogonality Relationships for Hermite Polynomials

Just like LDE this property turns out to be of vital importance not only in quantum mechanics but also in classical physics, where the polynomials form the basis of the associated Hermite functions, which in turn form part of the solution of the harmonic oscillator.

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \delta_{mn} 2^n n! \sqrt{\pi} \qquad \delta_{mn} = 0 \quad n \neq m$$
$$\delta_{mn} = 1 \quad n = m$$

Lets now prove the above. It's very important form examination point of view. We will prove it part by part. First we will show how it can be shown to zero then the other one ie $2^n n! \sqrt{\pi}$.

First for $\delta_{mn} = 0$ ie when $n \neq m$

By now we have come to know that $y = H_n(x)$ is a solution for LDE.

$$\frac{d^2 H_n}{dx^2} - 2x \frac{dH_n}{dx} + 2nH_n = 0$$

Now I will multiply the above expression by H_m keeping in mind that $H_m(x)$ is also a solution of HDE

$$H_m \frac{d^2 H_n}{dx^2} - 2xH_m \frac{dH_n}{dx} + 2nH_n H_m = 0 \tag{2.17}$$

Similarly for $H_m(x)$ the HDE is

$$\frac{d^2 H_m}{dx^2} - 2x \frac{dH_m}{dx} + 2mH_m = 0$$

Now I multiply the above expression with H_n to get the following

$$H_n \frac{d^2 H_m}{dx^2} - 2xH_n \frac{dH_m}{dx} + 2mH_n H_m = 0 \tag{2.18}$$

Now substracting equ^n(18) from equ^n(17) we get

$$\left[H_m \frac{d^2 H_n}{dx^2} - H_n \frac{d^2 H_m}{dx^2} \right] - 2x \left[H_m \frac{dH_n}{dx} - H_n \frac{dH_m}{dx} \right] + 2(n-m) H_n H_m = 0$$
$$e^{-x^2} \left[H_m \frac{d^2 H_n}{dx^2} - H_n \frac{d^2 H_m}{dx^2} \right] - 2xe^{-x^2} \left[H_m \frac{dH_n}{dx} - H_n \frac{dH_m}{dx} \right] + 2(n-m) e^{-x^2} H_n H_m = 0$$

Now we will do slight algebraic manipulation in the above so that some tricky algebra can be done and then we will integrate the above expression with respect to x from $-\infty$ to ∞ since within $[-\infty, \infty]$ the HDE converges for any value of x as x has all points as regular singular points

$$\int_{-\infty}^{\infty} \frac{d}{dx} \left[e^{-x^2} \left(H_m \frac{dH_n}{dx} - H_n \frac{dH_m}{dx} \right) \right] dx + 2(n-m) \int_{-\infty}^{\infty} e^{-x^2} H_n H_m dx = 0$$

Here you can do a simple calculation to show the first term in the left hand side is going to be zero within the stipulated limits. Thus we are left with the following

$$0 + 2(n-m) \int_{-\infty}^{\infty} e^{-x^2} H_n H_m dx = 0$$

Here we have two choices. Either $2(n-m) = 0$ or $\int_{-\infty}^{\infty} H_n H_m dx = 0$. But since $n \neq m$ therefore the integral part has to be zero. That means

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0 \qquad n \neq m} \tag{2.19}$$

Now for $\delta_{mn} = 1$ ie when $n = m$

In this case, the integration by parts technique wont work, since we cant count on the final integral being zero just like in case Legendre as well. Therfore to prove this one we will start with the generating function of HDE. There are other methods to prove that. But I find this one is easy and some of the principles of mathematics are previously known to you.

Ok we our generating function was

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2zx - z^2} \tag{2.20}$$

Similarly we still can have

$$\sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m = e^{2tx - t^2} \tag{2.21}$$

Now multiplying the above two equations we get

$$\begin{aligned}
e^{2zx-z^2}.e^{2tx-t^2} &= \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n.\sum_{m=0}^{\infty}\frac{H_m(x)}{m!}t^m \\
e^{2zx-z^2+2tx-t^2-x^2} &= e^{-x^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{H_n(x)H_m(x)}{n!m!}z^nt^m \\
&= e^{-x^2}\frac{H_n(x)H_m(x)}{n!m!}z^nt^m \\
&= e^{-x^2}\frac{H_n^2(x)}{(n!)^2}(zt)^n + e^{-x^2}\sum_{n\neq m}\sum_{n=0}^{\infty}\frac{H_n(x)H_m(x)}{n!m!}z^nt^m
\end{aligned}$$

Not convinced with the 2nd term in the RHS. Ok, so let me put it in the following. See your summation runs from $n = 0$ to $n = \infty$. So it covers all values of n. Let us take only two values of n as if the summation runs from $n = 0$ to $n = 1$. Then $\frac{H_n(x)}{n!}z^n$ will yeild the following

$$\sum_{n=0}^{1}\frac{H_n(x)}{n!}z^n = \frac{H_0}{0!}z^0 + \frac{H_1}{1!}z^1$$

Exactly in the similar note $\frac{H_m(x)}{m!}t^m$ will yeild the following if the summation runs from $n = 0$ to $n = 1$

$$\sum_{n=0}^{1}\frac{H_m(x)}{m!}t^m = \frac{H_0}{0!}t^0 + \frac{H_1}{1!}t^1$$

Now if I multiply the above two equation what we get is the following

$$\begin{aligned}
\sum_{n=0}^{1}\frac{H_n(x)}{n!}z^n.\sum_{n=0}^{1}\frac{H_m(x)}{m!}t^m &= \left(\frac{H_0}{0!}z^0 + \frac{H_1}{1!}z^1\right)\cdot\left(\frac{H_0}{0!}t^0 + \frac{H_1}{1!}t^1\right) \\
&= \frac{H_0}{0!}z^0\frac{H_0}{0!}t^0 + \frac{H_1}{1!}z^1.\frac{H_1}{1!}t^1 + \frac{H_1}{1!}z^1\frac{H_0}{0!}t^0 + \frac{H_0}{0!}z^0\frac{H_1}{1!}t^1
\end{aligned}$$

Now you see that the sum of the first two terms will give large$\sum_{n=0}^{1}\frac{H_n^2(x)}{(n!)^2}(zt)^n$ and the last two term will give $\sum_{n\neq m}\sum_{n=0}^{\infty}\frac{H_n(x)H_m(x)}{n!m!}z^nt^m$. If you are convinced let us now move forward.

Now we will be integrating the last equation after taking the product with respect to x over the limits $-\infty$ to ∞.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{2zx-z^2+2tx-t^2-x^2}dx &= \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n^2(x)}{(n!)^2}(zt)^n dx + \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n(x)H_m(x)}{n!m!}z^nt^m dx \\
\int_{-\infty}^{\infty} e^{2zx-z^2+2tx-t^2-x^2+2zt-2zt}dx &= \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n^2(x)}{(n!)^2}(zt)^n dx + \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n(x)H_m(x)}{n!m!}z^nt^m dx \\
e^{2zt}\int_{-\infty}^{\infty} e^{-(x-z-t)^2}dx &= \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n^2(x)}{(n!)^2}(zt)^n dx + 0 \qquad \int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = 0 \quad n\neq m
\end{aligned}$$

Recall your BSc.2nd sem where we did learned about Gamma function. That was $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$. And from that we moved on to calculate what was $\Gamma(\frac{1}{2})$. Well the value was $\sqrt{\pi}$ and the expression for that was $\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x^2}dx$. Therefore $\int_{-\infty}^{\infty} e^{-(x-z-t)^2}dx = \sqrt{\pi}$. Thus we arrive at

$$\begin{aligned}
e^{2zt}\sqrt{\pi} &= \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n^2(x)}{(n!)^2}(zt)^n dx \\
\sum_{n=0}^{\infty}\frac{(2zt)^n}{n!}\sqrt{\pi} &= \int_{-\infty}^{\infty} e^{-x^2}\frac{H_n^2(x)}{(n!)^2}(zt)^n dx
\end{aligned}$$

Now equating the co-eff. of(zt)^n on both side we get

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\frac{H_n^2(x)}{(n!)^2}dx &= 2^n\frac{\sqrt{\pi}}{n!} \\
\int_{-\infty}^{\infty} e^{-x^2}H_n^2(x)dx &= 2^n n!\sqrt{\pi}
\end{aligned}$$

Hence we have our orthoganality for same m and n as

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)dx = 2^n n!\sqrt{\pi} \qquad n = m} \tag{2.22}$$

2.6 Recurrence Relationships for Hermite Polynomials

In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. Thus a recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s).The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. To generate sequence basd on a recurrence relation, one must start with some initial values. If we denote the nth term in the sequence by x_n, such a recurrence relation is of the form $x_{n+1} = f(x_n)$. Sometimes it also refers to as **difference equation** to indicate some specific type of recurrence relation. In case Hermite differential equation we have four number of such equations. One can have even more. One just have play around the equations. These are in terms of different order polynomials or in terms of derivative of polynomial or a hybrid of both these. Let's investigate them.

First reccurence relation

Step I: Take the generating function (GF)

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

Step II: Differentiate the above expression w. r. to. z

$$\begin{aligned} e^{2zx-z^2}(2x-2z) =& n\frac{H_n(x)}{n!} z^{n-1} \\ \frac{H_n(x)}{n!} z^n (2x-2z) =& n\frac{H_n(x)}{n!} z^{n-1} \end{aligned}$$

Step III: Distribute the terms in either side

$$2x\frac{H_n(x)}{n!} z^n - 2\frac{H_n(x)}{n!} z^{n+1} = \frac{H_n(x)}{(n-1)!} z^{n-1}$$

Step IV: Now equate the co-efficient of z^n As this is a summation runs from 0 to ∞ therefore the series contains all the terms including the terms with different powers of z along with the polynomials. From there we have to pick the coefficients of z^n. Also I will be leaving (x) (ie the so called function of x as it is tiresome). Thus in a straight forward manner if we replace all the 'n-1's by either 'n's or 'n+1's except in the terms where there are already 'n-1' present, then we get

$$\begin{aligned} 2x\frac{H_n(x)}{n!} - 2\frac{H_{n-1}(x)}{(n-1)!} =& \frac{H_{n+1}(x)}{n!} \\ 2x\frac{H_n(x)}{n!} - 2n\frac{H_{n-1}(x)}{n!} =& \frac{H_{n+1}(x)}{n!} \\ 2xH_n(x) - 2nH_{n-1}(x) =& H_{n+1}(x) \end{aligned}$$

Thus the first recurrence relationship is $\boxed{H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)}$

Second reccurence relation

Step I: Take the generating function (GF)

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

Step II: Differentiate the above expression w. r. to. x

$$\begin{aligned} e^{2zx-z^2} 2z =& \frac{H'_n(x)}{n!} z^n \\ 2\frac{H_n(x)}{n!} z^{n+1} =& \frac{H'_n(x)}{n!} z^n \end{aligned}$$

Step III: Now equate the co-efficient of z^n

$$
\begin{aligned}
2\frac{H_{n-1}(x)}{(n-1)!} &= \frac{H'_n(x)}{n!} \\
2\frac{H_{n-1}(x)}{(n-1)!} &= \frac{H'_n(x)}{n(n-1)!} \\
2nH_{n-1}(x) &= H'_n(x)
\end{aligned}
$$

Thus the second recurrence relationship is $\boxed{2nH_{n-1}(x) = H'_n(x)}$

Third reccurence relation

Step I: Take the recurrence relationship I (RR-I)

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{2.23}$$

Step II: Replace the last term according to RR-II

$$
\begin{aligned}
H_{n+1}(x) = 2xH_n(x) - H'_n(x) \\
H'_n(x) - 2xH_n(x) + H_{n+1}(x) = 0
\end{aligned}
$$

Thus the third recurrence relationship is $\boxed{H'_n(x) - 2xH_n(x) + H_{n+1}(x) = 0}$

NOTE: Thus in order to proof RR-III you have to first proof RR-I and then RR-II. Of course that will depend upon the marks given in the question paper. Generally it will come as 4 marks question.

Fourth reccurence relation

Step I: Take the recurrence relationship III (RR-III)

$$H'_n(x) - 2xH_n(x) + H_{n+1}(x) = 0$$

Step II: Differentiate the above expression w. r. to. x

$$H''_n(x) - 2xH'_n(x) - 2H_n(x) + H'_{n+1}(x) = 0$$

Step III: Take RR-II

$$2nH_{n-1}(x) = H'_n(x)$$

Step IV: Replace n by $n+1$

$$2(n+1)H_n(x) = H'_{n+1}(x)$$

Step IV: Replace the last expression in step-II

$$
\begin{aligned}
H''_n(x) - 2xH'_n(x) - 2H_n(x) + 2(n+1)H_n(x) &= 0 \\
H''_n(x) - 2xH'_n(x) + 2nH_n(x) &= 0
\end{aligned}
$$

Thus the fourth recurrence relationship is $\boxed{H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0}$

NOTE: The fourth one is the HDE itself with $y = H_n(x)$. Thus in order to proof RR-IV you have to first proof RR-I. Then of course RR-II as we have used these to get RR-III. But then it will again depend upon the marks given in the question paper. See if you can do well. Wish you all the best.

2.7 LDE and HDE in a nutshell

Let us now summarize the main theme what we have learnt so far in these two differential equations. Let me put them in a tabular form so that it becomes a ready reference for you people.

Table 2.1: A comparative view of the LDE & HDE

Theme	The LDE	The HDE
The equation	$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$	$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$
Rodriques formula	$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^2, \qquad n \in N_0$	$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \qquad n \in N_0$
The Generating function	$(1-2xz+z^2)^{-\frac{1}{2}} = \sum z^n P_n(x)$	$e^{2zx-z^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n$
Orthogonality Relationship	$\int_{-1}^{1} P_n(x)P_m(x)dx = \delta_{mn}\frac{2}{2n+1}$ $\delta_{mn} = 0, n \neq m \quad \& \delta_{mn} = 1, n = m$	$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = \delta_{mn}2^n n!\sqrt{\pi}$ $\delta_{mn} = 0, n \neq m \quad \& \delta_{mn} = 1, n = m$
Recurrence Relationship	$nP_n = (2n-1)xP_{n-1} + (n-1)P_{n-2}$ $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$ $P_n' - xP_{n-1}' = nP_{n-1}$ $P_{n+1}' - P_{n-1}' = (2n+1)P_n$ $(x^2-1)P_n' = n(xP_n - P_{n-1})$ $(x^2-1)P_n' = (n+1)[P_{n+1} - xP_n]$	$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ $2nH_{n-1}(x) = H_n'(x)$ $H_n'(x) - 2xH_n(x) + H_{n+1}(x) = 0$ $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$